Free particle on SU(2) group manifold

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Abstract. In the present paper classical and quantum dynamics of a free particle on SU(2) group manifold is considered. Poisson structure of the dynamical system and commutation relations for generalized momenta are derived. Quantization is carried out and the eigenfunctions of the Hamiltonian are constructed in terms of coordinate free objects. SU(2)/U(1) coset model yielding after Hamiltonian reduction free particle on S^2 sphere is considered and Hamiltonian reduction of coset model is carried out on both classical and quantum level.

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1. Lagrangian description

The dynamics of a free particle on SU(2) group manifold is described by the Lagrangian

$$L = \langle g^{-1}\dot{g}g^{-1}\dot{g}\rangle \tag{1}$$

where $g \in SU(2)$ and $\langle\ \rangle$ denotes the normalized trace

$$\langle \cdot \rangle = -\tfrac{1}{2}\mathrm{Tr}(\cdot) \tag{2}$$

which defines a scalar product in su(2) algebra. This Lagrangian gives rise to equations of motion

$$\frac{d}{dt}g^{-1}\dot{g} = 0 \tag{3}$$

that describe dynamics of particle on group manifold. Also, one can notice that it has SU(2) "right" and SU(2) "left" symmetry. It means that it is invariant under the following transformations

$$\begin{aligned} g &\to h_1 g \\ g &\to g h_2 \end{aligned} \tag{4}$$

where $h_1, h_2 \in SU(2)$

According to the Noether's theorem these symmetries lead to the matrix valued conserved quantities

$$C = g^{-1}\dot{g} \qquad \frac{d}{dt}C = 0 \tag{5}$$

and

$$S = \dot{g}g^{-1} \qquad \frac{d}{dt}S = 0 \tag{6}$$

To construct integrals of motion out of C and S let us introduce the basis of su(2) algebra — three matrices:

$$T_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{7}$$

The elements of su(2) are traceless anti-hermitian matrices, and any $A \in su(2)$ can be parameterized in the following way

$$A = A^n T_n \qquad n = 1, 2, 3 \tag{8}$$

Scalar product

$$AB = \langle AB \rangle = -\tfrac{1}{2}\mathrm{Tr}(AB) \tag{9}$$

ensures that

$$A^n = \langle AT_n \rangle \qquad (\langle T_n T_m \rangle = \delta_{nm}) \tag{10}$$

Now we can introduce six functions

$$\begin{aligned} C_n &= \langle T_n C \rangle \qquad n = 1, 2, 3 \qquad C = C^n T_n \\ S_n &= \langle T_n S \rangle \qquad n = 1, 2, 3 \qquad S = S^n T_n \end{aligned} \tag{11}$$

which are integrals of motion.

Conservation of C and S leads to general solution of Euler-Lagrange equations

$$\begin{aligned} &\frac{d}{dt}g^{-1}\dot{g} = 0 \qquad \Rightarrow \qquad g^{-1}\dot{g} = \mathrm{const} \\ &g = e^{Ct}g(0) \end{aligned} \tag{12}$$

These are well known geodesics on Lie group.

2. Hamiltonian description

Working in a first order Hamiltonian formalism we can construct new Lagrangian which is equivalent to the initial one

$$\Lambda = \langle C(g^{-1}\dot{g} - v) \rangle + \tfrac{1}{2}\langle v^2 \rangle \tag{13}$$

in sense that variation of C provides

$$g^{-1}\dot{g} = v \tag{14}$$

and Λ reduces to L. Variation of v gives C = v and therefore we can rewrite equivalent Lagrangian Λ in terms of C and g variables

$$\Lambda = \langle Cg^{-1}\dot{g}\rangle - ½\,\langle C^2\rangle \tag{15}$$

where function

$$H = ½\langle C^2\rangle \tag{16}$$

plays the role of Hamiltonian and one-form $\langle Cg^{-1}dg\rangle$ is a symplectic potential θ. External differential of θ is the symplectic form

$$\omega = d\theta = -\langle g^{-1}dg \wedge dC\rangle - \langle Cg^{-1}\, dg \wedge g^{-1}dg\rangle \tag{17}$$

that determines Poisson brackets, the form of Hamilton's equation and provides isomorphism between vector fields and one-forms

$$X \quad \rightarrow \quad i_X\omega \tag{18}$$

For any smooth SU(2) valued smooth function $f \in SU(2)$ one can define Hamiltonian vector field X_f by

$$i_{X_f}\omega = -\, df \tag{19}$$

where $i_X\omega$ denotes the contraction of X with ω. According to its definition Poisson bracket of two functions is

$$\{f\, ,\, g\} = L_{X_f}g = i_{X_f}dg = \omega(X_f\, ,\, X_g) \tag{20}$$

where $L_{X_f}g$ denotes Lie derivative of g with respect to vector filed X_f. The skew symmetry of ω provides skew symmetry of Poisson bracket.

Hamiltonian vector fields that correspond to C_n, S_m and g functions are

$$\begin{aligned} X_n &= X_{C_n} = ([C\, ,T_n]\, ,\, gT_n) \\ Y_m &= X_{S_m} = ([C\, ,\, gT_mg^{-1}]\, ,\, T_mg\,) \end{aligned} \tag{21}$$

and give rise to the following commutation relations

$$\begin{aligned} \{S_n\, ,\, S_m\} &= -\, 2\varepsilon_{nm}{}^{k}\, S_k \\ \{C_n\, ,\, C_m\} &= 2\varepsilon_{nm}{}^{k}\, C_k \\ \{C_n\, ,\, S_m\} &= 0 \\ \{C_n\, ,\, g\} &= gT_n \\ \{S_m\, ,\, g\} &= T_mg \end{aligned} \tag{22}$$

The results are natural. C and S that correspond respectively to the "right" and "left" symmetry commute with each other and independently form su(2) algebras. Now knowing Poisson bracket structure one can write down Hamilton's equations

$$\dot{g} = \{H\, ,\, g\} = gR \tag{23}$$

$$\dot{C} = \{H\, ,\, C\} = 0 \tag{24}$$

3. Quantization

Let's introduce operators

$$\hat{C}_n = \frac{i}{2}L_{X_n} \tag{25}$$

$$\hat{S}_m = -\frac{i}{2}L_{Y_m} \tag{26}$$

They act on the square integrable functions (see Appendix A) on SU(2) and satisfy quantum commutation relations

$$[\hat{S}_n , \hat{S}_m] = i\varepsilon_{nm}{}^{k}\,\hat{S}_k \tag{27}$$

$$[\hat{C}_n , \hat{C}_m] = i\varepsilon_{nm}{}^{k}\,\hat{C}_k \tag{28}$$

$$[\hat{C}_n , \hat{S}_m] = 0 \tag{29}$$

The Hamiltonian is defined as

$$\hat{H} = \hat{C}^2 = \hat{S}^2 \tag{30}$$

and the complete set of observables that commute with each other is

$$\hat{H}, \qquad \hat{C}_a, \qquad \hat{S}_b \tag{31}$$

with some fixed a and b. Using a simple generalization of a well known algebraic construction (see Appendix B) one can check that the eigenvalues of the quantum observables $\hat{H}$, $\hat{C}_a$ and $\hat{S}_b$ have the form

$$\hat{H}\psi_{jsc} = j(j + 1)\psi_{jsc} \tag{32}$$

where j takes positive integer and half integer values

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \ldots \tag{33}$$

$$\hat{C}_a\psi_{jsc} = c\psi_{jsc} \tag{34}$$

$$\hat{S}_b\psi_{jsc} = s\psi_{jsc} \tag{35}$$

with c and s taking values in the following range

$$-j, -j + 1, \ldots , j - 1, j \tag{36}$$

Further we construct the corresponding eigenfunctions ψ_{jsc}. The first step of this construction is to note that the function $\langle Tg \rangle$ where $T = (1 + iT_a)(1 + iT_b)$ is an eigenfunction of $\hat{H}$, $\hat{C}_a$ and $\hat{S}_b$ with eigenvalues ¾, ½, ½ respectively. Proof of this proposition is straightforward. Using $\langle Tg \rangle$ one can construct the complete set of eigenfunctions of $\hat{H}$, $\hat{C}_a$ and $\hat{S}_b$ operators

$$\psi_{jsc} = \hat{S}_-^{\,j-s}\hat{C}_-^{\,j-c}\langle Tg \rangle^{2j} \tag{37}$$

in the manner described in Appendix B.

4. Free particle on S^2 as a SU(2)/U(1) coset model

Free particle on 2D sphere can be obtained from our model by gauging U(1) symmetry. In other words let's consider the following local gauge transformations

$$g \quad \to \quad h(t)g \tag{38}$$

Where $h(t) \in U(1) \subset SU(2)$ is an element of U(1). Without loss of generality we can take

$$h = e^{\beta(t)T_3} \tag{39}$$

Since T_3 is antihermitian $h(t) \in U(1)$ and since h(t) depends on t Lagrangian

$$L = \langle g^{-1}\dot{g}g^{-1}\dot{g}\rangle \tag{40}$$

is not invariant under (38) local gauge transformations.

To make (40) gauge invariant we should replace time derivative with covariant derivative

$$\frac{d}{dt}g \quad \to \quad \nabla g = \left(\frac{d}{dt} + B\right)g \tag{41}$$

where B can be represented as follows

$$B = bT_3 \in su(2) \tag{42}$$

with transformation rule

$$B \quad \to \quad hBh^{-1} - \frac{dh}{dt}h^{-1} \tag{43}$$

or in terms of b variable

$$b \quad \to \quad b - \frac{d\beta}{dt} \tag{44}$$

The new Lagrangian

$$L_G = \langle g^{-1}\nabla g g^{-1}\nabla g\rangle \tag{45}$$

is invariant under (38) local gauge transformations. But this Lagrangian as well as every gauge invariant Lagrangian is singular. It contains additional non-physical degrees of freedom. To eliminate them we should eliminate B using Lagrange equations

$$\frac{\partial L_G}{\partial B} \quad \to \quad b = -\langle \dot{g}g^{-1}T_3\rangle \tag{46}$$

put it back in (45) and rewrite last obtained Lagrangian in terms of gauge invariant variables.

$$L_G = \langle (g^{-1}\dot{g} - S_3T_3)^2 \rangle \tag{47}$$

It's obvious that the following

$$Z = g^{-1}T_3 g \in su(2) \tag{48}$$

element of su(2) algebra is gauge invariant. Since $Z \in su(2)$ it can be parameterized as follows

$$Z = z^a T_a \tag{49}$$

where z^a are real functions on SU(2)

$$z_a = \langle Z T_a \rangle \tag{50}$$

So we have three gauge invariant variables z^a (a = 1, 2, 3) but it's easy to check that only two of them are independent. Indeed

$$\langle Z^2 \rangle = \langle g^{-1}T_3 g g^{-1} T_3 g \rangle = \langle T_3{}^2 \rangle = 1 \tag{51}$$

otherwise

$$\langle Z^2 \rangle = \langle z^a T_a z^b T_b \rangle = z^a z_a \tag{52}$$

So configuration space of SU(2)/U(1) coset model is sphere. By direct calculations one can check that after being rewritten in terms of gauge invariant variables L_G takes the form

$$L_G = ¼\langle Z^{-1}\dot{Z} Z^{-1}\dot{Z} \rangle \tag{53}$$

This Lagrangian describes free particle on the sphere. Indeed, since $Z = z^a T_a$ it's easy to show that

$$L_G = ¼\langle Z^{-1}\dot{Z} Z^{-1}\dot{Z} \rangle = ¼\langle Z\dot{Z}Z\dot{Z} \rangle = ½\dot{z}^a \dot{z}_a \tag{54}$$

So SU(2)/U(1) coset model describes free particle on S^2 manifold.

5. Quantization of the coset model.

Working in a first order Hamiltonian formalism one can introduce equivalent Lagrangian

$$\Lambda_G = \langle C(g^{-1}\dot{g} - u) \rangle + ½ \langle (u + g^{-1}Bg)^2 \rangle \tag{55}$$

variation of u provides

$$C = u + g^{-1}Bg \tag{56}$$
$$u = C - g^{-1}Bg$$

Rewriting Λ_G in terms of C and g leads to

$$\Lambda_G = \langle Cg^{-1}\dot{g}\rangle - ½\langle C^2\rangle - \langle BgCg^{-1}\rangle = \langle Cg^{-1}\dot{g}\rangle - ½\langle C^2\rangle - b\langle gCg^{-1}T_3\rangle = \langle Cg^{-1}\dot{g}\rangle - ½\langle C^2\rangle - bS_3 \quad (57)$$

Due to the gauge invariance of Λ_G we obtain constrained Hamiltonian system, where $\langle Cg^{-1}dg\rangle$ is symplectic potential, function

$$H = ½\langle C^2\rangle \quad (58)$$

plays the role of Hamiltonian and b is a Lagrange multiple leading to the first class constrain

$$\phi = \langle gCg^{-1}T_3\rangle = \langle ST_3\rangle = S_3 = 0 \quad (59)$$

So coset model is equivalent to the initial one with (59) constrain. Using technique of the constrained quantization, instead of quantizing coset model we can subject quantum model that corresponds to the free particle on SU(2), to the following operator constrain

$$\hat{S}_3 | \psi\rangle = 0 \quad (60)$$

Hilbert space of the initial system, that is linear span of

$$\psi_{jcs} \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \quad (61)$$

wave functions, reduces to the linear span of

$$\psi_{jc0} \qquad j = 0, 1, 2, 3, \ldots \quad (62)$$

wave functions. Indeed, $\hat{S}_3\psi_{jcs} = 0$ implies s = 0, and if s = 0 then j is integer. Thus c takes $-j, -j+1, \ldots, j-1, j$ integer values only. Wave functions ψ_{jcs} rewriten in terms of gauge invariant variables up to a constant multiple should coincide with well known spherical harmonics

$$\psi_{jc0} \sim J_{jc} \quad (63)$$

One can check the following

$$\psi_{jc0} \sim \hat{S}_-^{\,j}\hat{C}_-^{\,j-c}\langle Tg\rangle^{2j} \sim \hat{C}_-^{\,j-c}\langle T_+g^{-1}T_3g\rangle^j \sim \hat{C}_-^{\,j-c}\sin^j\theta e^{ij\theta} \sim \hat{C}_-^{\,j-c}J_{jj} \sim J_{jc} \quad (64)$$

This is an example of using large initial model in quantization of coset model.

6. Appendix A

Scalar product in Hilbert space is defined as follows

$$\langle\psi_1 | \psi_2\rangle = \int_{SU(2)} \prod_{a=1}^{3} \langle g^{-1}dgT_a\rangle(\psi_1)^{\dagger}\psi_2 \quad (65)$$

It's easy to prove that under this scalar product operators $\hat{C}_n$ and $\hat{S}_m$ are hermitian. Indeed

$$\langle\psi_1|\hat{C}_n\psi_2\rangle = \int_{SU(2)} \prod_{a=1}^{3} \langle g^{-1}dgT_a\rangle(\psi_1)^\dagger\left(\frac{i}{2}L_{X_n}\psi_2\right) = \int_{SU(2)} \prod_{a=1}^{3} \langle g^{-1}dgT_a\rangle\left(\frac{i}{2}L_{X_n}\psi_1\right)^\dagger \psi_2 \tag{66}$$

Where integration by part has been used and the additional term coming from measure

$$\prod_{a=1}^{3} \langle g^{-1}dgT_a\rangle \tag{67}$$

vanished since

$$L_{X_n}\langle g^{-1}dgT_a\rangle = 0 \tag{68}$$

For more transparency one can introduce the following parameterization of SU(2). For any $g \in SU(2)$.

$$g = e^{q^aT_a} \tag{69}$$

Then the symplectic potential takes the form

$$\langle Cg^{-1}dg\rangle = C_adq^a \tag{70}$$

and scalar product becomes

$$\langle\psi_1|\psi_2\rangle = \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} d^3q(\psi_1)^\dagger\psi_2 \tag{71}$$

that coincides with (65) because of

$$dq_a = \langle g^{-1}dg\ T_a\rangle \tag{72}$$

7. Appendix B

Without loss of generality we can take $\hat{H}$, $\hat{S}_3$ and $\hat{C}_3$ as a complete set of observables. Assuming that operators $\hat{H}$, $\hat{S}_3$ and $\hat{C}_3$ have at least one common eigenfunction

$$\hat{H}\psi = E\psi \qquad \hat{C}_3\psi = c\psi \qquad \hat{S}_3\psi = s\psi \tag{73}$$

it is easy to show that eigenvalues of $\hat{H}$ are non-negative $E \geq 0$ and conditions

$$E - c^2 \geq 0 \qquad E - s^2 \geq 0 \tag{74}$$

are satisfied. Indeed, operators $\hat{C}$ and $\hat{S}$ are selfadjoint so

$$\langle\psi|\hat{H}|\psi\rangle = \langle\psi|\hat{C}^2|\psi\rangle = \langle\psi|\hat{C}_a\hat{C}^a|\psi\rangle = \langle\psi|(\hat{C}_a)^\dagger\hat{C}^a|\psi\rangle = \langle\hat{C}_a\psi|\hat{C}^a\psi\rangle = ?\hat{C}_a\psi? \geq 0 \tag{75}$$

To prove (74) we shall consider $\hat{C}_1{}^2 + \hat{C}_2{}^2$ and $\hat{S}_1{}^2 + \hat{S}_2{}^2$ operators

$$\langle\psi|\hat{C}_1{}^2 + \hat{C}_2{}^2|\psi\rangle = ?\hat{C}_1\,\psi? + ?\hat{C}_2\,\psi? \geq 0 \tag{76}$$

and

$$\langle\psi|\hat{C}_1{}^2 + \hat{C}_2{}^2|\psi\rangle = \langle\psi|\hat{H} - \hat{C}_3{}^2|\psi\rangle = (E - c^2)\langle\psi|\psi\rangle \tag{77}$$

thus $E - c^2 \geq 0$.

Now let's introduce new operators

$$\hat{C}_+ = i\hat{C}_1 + \hat{C}_2 \qquad \hat{C}_- = i\hat{C}_1 - \hat{C}_2 \tag{78}$$

$$\hat{S}_+ = i\hat{S}_1 + \hat{S}_2 \qquad \hat{S}_- = i\hat{S}_1 - \hat{S}_2 \tag{79}$$

These operators are not selfadjoint, but $(\hat{C}_-)^\dagger = \hat{C}_+$ and $(\hat{S}_-)^\dagger = \hat{S}_+$ and they fulfill the following commutation relations

$$[\hat{C}_\pm , \hat{C}_3] = \pm\,\hat{C}_\pm \qquad [\hat{S}_\pm , \hat{S}_3] = \pm\,\hat{S}_\pm \tag{80}$$

$$[\hat{C}_+ , \hat{C}_-] = 2\hat{C}_3 \qquad [\hat{S}_+ , \hat{S}_-] = 2\hat{S}_3 \tag{81}$$

$$[\hat{C}_\bullet , \hat{S}_\bullet] = 0 \tag{82}$$

where • takes values +, −, 3 using these commutation relations it is easy to show that if $\psi_{\lambda cs}$ is eigenfunction of $\hat{H}$, $\hat{S}_3$ and $\hat{C}_3$ with corresponding eigenvalues :

$$\begin{aligned} \hat{H}\psi_{\lambda cs} &= \lambda\psi_{\lambda cs} \\ \hat{S}_3\psi_{\lambda cs} &= s\psi_{\lambda cs} \\ \hat{C}_3\psi_{\lambda cs} &= c\psi_{\lambda cs} \end{aligned} \tag{83}$$

then $\hat{C}_\pm\psi_{\lambda cs}$ and $\hat{S}_\pm\psi_{\lambda cs}$ are the eigenfunctions with corresponding eigenvalues λ, s ± 1, c and λ , s, c ± 1. Consequently using $\hat{C}_\pm$, $\hat{S}_\pm$ operators one can construct a family of eigenfunctions with eigenvalues

$$\begin{aligned} &c,\ c \pm 1,\ c \pm 2,\ c \pm 3,\ \ldots \\ &s,\ s \pm 1,\ s \pm 2,\ s \pm 3,\ \ldots \end{aligned} \tag{84}$$

but conditions (74) give restrictions on a possible range of eigenvalues. Namely we must have

$$\begin{aligned} \lambda - c^2 &\geq 0 \\ \lambda - s^2 &\geq 0 \end{aligned} \tag{85}$$

In other words, in order to interrupt (84) sequences we must assume

$$\hat{S}_+ \psi_{\lambda cj} = 0 \qquad \hat{S}_- \psi_{\lambda c, -j} = 0$$
$$\hat{C}_+ \psi_{\lambda ks} = 0 \qquad \hat{C}_- \psi_{\lambda, -ks} = 0 \tag{86}$$

for some j and k, therefore s and c could take only the following values

$$-j, -j+1, \dots, j-1, j$$
$$-k, -k+1, \dots, k-1, k \tag{87}$$

The number of values is 2j + 1 and 2k + 1 respectively. Since number of values should be integer, j and k should take integer or half integer values

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$$
$$k = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots \tag{88}$$

Now using commutation relations we can rewrite $\hat{H}$ in terms of $\hat{C}_\pm$, $\hat{C}_3$ operators

$$\hat{H} = \hat{C}_+ \hat{C}_- + \hat{C}_3^{\,2} + \hat{C}_3 \tag{89}$$

and it is clear that j = k and $\lambda = j(j+1) = k(k+1)$

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