Role of non-Noether symmetry in integrability of dispersiveless long wave system

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Abstract. We show that infinite sequence of conserved quantities and bi-Hamiltonian structure of DLW hierarchy of integrable models are related to the non-Noether symmetry of dispersiveless water wave system.

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Symmetries play essential role in dynamical systems, because they usually simplify analysis of evolution equations and often provide quite elegant solution of problems that otherwise would be difficult to handle. In the present paper we show how knowing just single generator of non-Noether symmetry one can construct infinite involutive sequence of conserved quantities and bi-Hamiltonian structure of one of the remarkable integrable models — dispersiveless long wave system. In fact among nonlinear partial differential equations that describe propagation of waves in shallow water there are many interesting integrable models. And most of them seem to have non-Noether symmetries leading to the infinite sequence of conservation laws and bi-Hamiltonian realization of these equations. In dispersiveless long wave system such a symmetry appears to be local, that in some sense simplifies and investigation of its properties and calculations of conserved quantities.

Evolution of dispersiveless long wave system is governed by the following set of nolinear partial differential equations

$$
\begin{aligned}
v_t &= v_x w + v w_x \\
w_t &= v_x + w w_x
\end{aligned}
\tag{1}
$$

Each symmetry of this system must satisfy linear equation

$$
\begin{aligned}
E(v)_t &= (wE(v))_x + (vE(w))_x \\
E(w)_t &= E(v)_x + (wE(w))_x
\end{aligned}
\tag{2}
$$

obtained by substituting infinitesimal transformations

$$
\begin{aligned}
v &\to v + aE(v) + O(a^2) \\
w &\to w + aE(w) + O(a^2)
\end{aligned}
\tag{3}
$$

into equations of motion (1) and grouping first order (in a) terms. One of the solutions of this equation yields the following symmetry of dispersiveless water wave system

$$E(v) = 4vw + 2x(vw)_x + 3t(v^2 + vw^2)_x \qquad (4)$$
$$E(w) = w^2 + 4v + 2x(ww_x + v_x) + t(6vw + w^3)_x$$

and it is remarkable that this symmetry is local in sense that E(u) in point x depends only on u and its derivatives evaluated in the same point (this is not the case in Korteweg-de Vriez, modified Korteweg-de Vriez and nonlinear Schrödinger equations where similar symmetries appear to be non local [3])

Before we proceed let us note that dispersive water wave system is actually infinite dimensional Hamiltonian dynamical system. Assuming that u, v and w fields are subjected to zero boundary conditions

$$v(\pm \infty) = w(\pm \infty) = 0 \qquad (5)$$

it is easy to verify that equations (1) can be represented in Hamiltonian form

$$v_t = \{h , v\} \qquad (6)$$
$$w_t = \{h , w\}$$

with Hamiltonian equal to

$$h = -\tfrac{1}{2} \int_{-\infty}^{+\infty} (vw^2 + v^2)dx \qquad (7)$$

and Poisson bracket defined by the following Poisson bivector field

$$W = \int_{-\infty}^{+\infty} \left(\frac{\delta}{\delta v} \wedge \left(\frac{\delta}{\delta w} \right)_x \right) dx \qquad (8)$$

Now using our symmetry that appears to be non-Noether, one can calculate second Poisson bivector field involved in the bi-Hamiltonian realization of dispersiveless long wave system

$$\hat{W} = [E , W] = -2 \int_{-\infty}^{+\infty} \left(v\frac{\delta}{\delta v} \wedge \left(\frac{\delta}{\delta v} \right)_x + w\,\frac{\delta}{\delta v} \wedge \left(\frac{\delta}{\delta w} \right)_x + \frac{\delta}{\delta w} \wedge \left(\frac{\delta}{\delta w} \right)_x \right) dx \qquad (9)$$

Note that $\hat{W}$ give rise to the second Hamiltonian realization of the model

$$v_t = \{\hat{h} , v\}_* \qquad (10)$$
$$w_t = \{\hat{h} , w\}_*$$

where

$$\hat{h} = -\tfrac{1}{2}\int_{-\infty}^{+\infty} vw\,dx \qquad (11)$$

and $\{\ ,\ \}_*$ is Poisson bracket defined by bivector field $\hat{W}$.

Now let us pay attention to conservation laws. By integrating third equation of dispersive water wave system (1) it is easy to show that

$$J^{(0)} = \int_{-\infty}^{+\infty} w\,dx \qquad (12)$$

is conservation law. Using non-Noether symmetry one can construct other conservation laws by taking Lie derivative of $J^{(0)}$ along the generator of symmetry and in this way entire infinite sequence of conservation laws of dispersive water wave system can be reproduced

$$J^{(0)} = \int_{-\infty}^{+\infty} w\,dx \qquad (13)$$

$$J^{(1)} = L_E J^{(0)} = 2\int_{-\infty}^{+\infty} v\,dx$$

$$J^{(2)} = L_E J^{(1)} = (L_E)^2 J^{(0)} = 4\int_{-\infty}^{+\infty} vw\,dx$$

$$J^{(3)} = L_E J^{(2)} = (L_E)^3 J^{(0)} = 12\int_{-\infty}^{+\infty} (vw^2 + v^2)\,dx$$

$$J^{(4)} = L_E J^{(3)} = (L_E)^4 J^{(0)} = 48\int_{-\infty}^{+\infty} (3v^2 w + vw^3)\,dx$$

$$J^{(n)} = L_E J^{(n-1)} = (L_E)^n J^{(0)}$$

So as we see non-Noether symmetry (4) naturally leads to infinite sequence of conserved quantities and second Hamiltonian realization of dispersiveless water wave system.

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References

[1] G. Bluman, S. Kumei, *Symmetries and differential equations*, Springer-Verlag, New York, 1989

[2] G. Chavchanidze, *Non-Noether symmetries and their influence on phase space geometry*, J. Geom. Phys. 48, 190-202, 2003

[3] G. Chavchanidze, *Non-Noether symmetries in Hamiltonian dynamical systems*, math-ph/0405003, 2004

[4] P. Olver, *Applications of Lie groups to differential equations*, GTM 107, Springer Verlag, New York, 1986